Journal of Geometry and Physics 37 (2001) 201–215

Spinor-valued and Clifford algebra-valued harmonic polynomials

Yasushi Homma

Department of Mathematical Sciences, Waseda University, 3-4-1 Ohkubo, Shinjuku-ku, Tokyo 169-8555, Japan

Received 27 July 1999; received in revised form 17 April 2000

Abstract

We give decompositions of the spinor-valued and the Clifford algebra-valued harmonic polynomials on $\mathbf{R}^n$. In order to do so, we consider some differential complexes and show that these are exact. As a corollary, we have Poincaré lemma for harmonic polynomials. Besides, we prove that each component of the decompositions is an irreducible representation space with respect to $Spin(n)$. © 2001 Elsevier Science B.V. All rights reserved.

MSC: 43A85; 43A90; 58J05

Subj. Class.: Spinors; Twistors

Keywords: Clifford-valued polynomials; $Spin(n)$-modules; Invariant operators

1. Introduction

Spherical harmonic polynomials or spherical harmonics as building blocks for analysis on the sphere are traditionally an indispensable tool in mathematical physics. Recently, the interest was shifted from functions space on the sphere to spaces of sections of natural bundles. These spaces are representation spaces of $Spin(n)$ and their irreducible components are often given by polynomial solutions of invariant differential operators. For example, spaces of spinor-valued functions on the sphere are spaces of the so-called spherical monogenics. They are spinor-valued polynomial solutions of the Dirac equations on $\mathbf{R}^n$ [6,8,10,14]. The Clifford algebra-valued fields and other examples are studied in [4,5,7–9,11]. In this paper, we give a new approach to analyze the Clifford algebra-valued or the exterior algebra-valued fields on the sphere.

The space of functions on S^{n-1} is isomorphic to the space of harmonic polynomials on $\mathbf{R}^n$. Similarly, we can regard the sections of the spinor bundle (resp. the Clifford bundle)

E-mail address: homma@gm.math.waseda.ac.jp (Y. Homma).

PII: S0393-0440(00)00045-0

as the spinor-valued (resp. the Clifford algebra-valued) harmonic polynomials. Let H^q be the space of the harmonic polynomials with degree q on $\mathbf{R}^n$. We consider $\sum H^q \otimes W_n$ and $\sum H^q \otimes \mathbf{C}l_n$, where W_n is the space of spinors and $\mathbf{C}l_n$ is the Clifford algebra. Trautman [14] gives a geometrical decomposition of $H^q \otimes W_n$ by using the Dirac operator D and the algebraic operator x, where the important tool to analyze $H^q \otimes W_n$ is the spinor complex $(H^* \otimes W_n, D)$. We consider an analogue of the spinor complex for the Clifford algebra-valued harmonic polynomials. Since $\mathbf{C}l_n$ is isomorphic to the exterior algebra $\sum \Lambda^p_{\mathbf{C}}(\mathbf{R}^n)$, we use the differential operators d and d* instead of the Dirac operator and have the de Rham complex $(H^* \otimes \Lambda^*_{\mathbf{C}}(\mathbf{R}^n), \mathrm{d})$ for harmonic polynomials. To show the exactness of this complex is more complicated than the spinor case. So we give a geometric decomposition of $H^q \otimes \Lambda^p_{\mathbf{C}}(\mathbf{R}^n)$ first. Then we have the exactness of the de Rham complex for harmonic polynomials. Since the operator d and d* are invariant operators, we prove that each component of our decompositions is irreducible with respect to $Spin(n)$.

Sections 2 and 3 are preliminaries. In Section 2, we describe the Clifford algebra and some representations of $Spin(n)$. In Section 3, we have the Clifford bundle and the spinor bundle on the sphere and give trivializations of these bundles. Then, we can regard the sections of these bundle as the spinor-valued and the Clifford algebra-valued harmonic polynomials on $\mathbf{R}^n$. In Section 4, we present Trautman's theory for the spinor-valued harmonic polynomials. Section 5 is the main of this paper. We study the Clifford algebra-valued or the exterior algebra-valued harmonic polynomials. We have the differential operators d and d* and the algebraic operators $i(x)$ and $-x_\wedge$ for them. By using these operators, we decompose $H^q \otimes \Lambda^p_{\mathbf{C}}(\mathbf{R}^n)$ and show that the de Rham complex for harmonic polynomials is exact. In Section 6, we present some results for the representation of the Lie algebra $\mathfrak{spin}(n) \otimes \mathbf{C} \simeq \mathfrak{so}(n, \mathbf{C})$ by using the Clifford algebra. In Section 7, we verify that our geometrical decompositions correspond to the irreducible ones with respect to $Spin(n)$.

2. The Clifford algebra and the spinor space

Let $\mathbf{R}^n$ be the n-dimensional Euclidean space with the orthonormal basis $\{e_k\}_{k=1}^n$. Then, we have the complex Clifford algebra $\mathbf{C}l_n$, where the relations among $\{e_k\}_{k=1}^n$ are given by

$$e_k e_l + e_l e_k = -2\delta_{kl}. \tag{2.1}$$

The following vector space isomorphism is well known:

$$\mathbf{C}l_n \ni e_{k_1} e_{k_2} \cdots e_{k_p} \mapsto e_{k_1} \wedge e_{k_2} \wedge \cdots \wedge e_{k_p} \in \sum_p \Lambda^p_{\mathbf{C}}(\mathbf{R}^n), \tag{2.2}$$

where $\sum \Lambda^p_{\mathbf{C}}(\mathbf{R}^n)$ is the vector space of the complex exterior algebra associated to $\mathbf{R}^n$. Besides, $\mathbf{C}l_n$ has the decomposition to the even and the odd parts, $\mathbf{C}l_n = \mathbf{C}l_n^0 \oplus \mathbf{C}l_n^1$. Here $\mathbf{C}l_n^i$ is isomorphic to $\sum_k \Lambda^{2k+i}_{\mathbf{C}}(\mathbf{R}^n)$.

We shall prepare two homomorphisms on the Clifford algebras [1]. First, we know that the sub-algebra $\mathbf{C}l_n^0$ is isomorphic to $\mathbf{C}l_{n-1}$ as an algebra by the map $j : \mathbf{C}l_{n-1} \xrightarrow{\simeq} \mathbf{C}l_n^0$ which extends the map $j(e_k) = e_n e_k$ for e_k in $\mathbf{R}^{n-1}$. Next, from the natural inclusion

$i : \mathbf{R}^{n-1} \to \mathbf{R}^n$, we have its extension, the map $i : \mathbf{C}l_{n-1} \to \mathbf{C}l_n$. Here, we remark that the map i coincides with j on $\mathbf{C}l_{n-1}^0$.

Now, we investigate the spinor representation (Δ_n, W_n) and adjoint representation $(\mathrm{Ad}_n, \mathbf{C}l_n)$ of $Spin(n)$, where $Spin(n)$ is the spin group in $\mathbf{C}l_n$. The spinor representation (Δ_n, W_n) is the restriction of an irreducible $\mathbf{C}l_n$-module to $Spin(n)$. For $n = 2m$, (Δ_{2m}, W_{2m}) decomposes as the direct sum of two inequivalent irreducible representations $(\Delta_{2m}^+, W_{2m}^+)$ and $(\Delta_{2m}^-, W_{2m}^-)$. If we restrict the spinor representation of $Spin(n)$ to its subgroup $Spin(n-1)$, we have spinor representations of $Spin(n-1)$ [6]:

$$(\Delta_{2m}^{\pm}|_{Spin(2m-1)}, W_{2m}^{\pm}) \simeq (\Delta_{2m-1}, W_{2m-1}), \tag{2.3}$$

$$(\Delta_{2m+1}|_{Spin(2m)}, W_{2m+1}) \simeq (\Delta_{2m}, W_{2m}). \tag{2.4}$$

These isomorphisms are important to trivialize the spinor bundle on S^{n-1}.

The adjoint representation $(\mathrm{Ad}_n, \mathbf{C}l_n)$ is given by

$$Spin(n) \times \mathbf{C}l_n \ni (g, \psi) \mapsto \mathrm{Ad}_n(g)\psi = g\psi g^{-1} \in \mathbf{C}l_n. \tag{2.5}$$

Under the isomorphism $\mathbf{C}l_n \simeq \sum \Lambda_{\mathbf{C}}^p(\mathbf{R}^n)$, the vector space $\Lambda_{\mathbf{C}}^p(\mathbf{R}^n)$ is invariant. Hence, so is $\mathbf{C}l_n^i$ for $i = 0, 1$. We denote the representation of $Spin(n)$ on $\mathbf{C}l_n^i$ by Ad_n^i. The following lemma is an analogue of (2.3) and (2.4) for the Clifford case.

Lemma 2.1. *If we think of $Spin(n-1)$ as a subgroup of $Spin(n)$ by the map i, then $\Lambda_{\mathbf{C}}^p(\mathbf{R}^n)$ is isomorphic to $\Lambda_{\mathbf{C}}^p(\mathbf{R}^{n-1}) \oplus \Lambda_{\mathbf{C}}^{p-1}(\mathbf{R}^{n-1})$ as a representation space of $Spin(n-1)$. In particular*, $\mathrm{Ad}_n^i|_{Spin(n-1)}$ *is equivalent to* Ad_{n-1}.

Proof. For $p = 2s$, restricting the domain of the map j to $\Lambda_{\mathbf{C}}^p(\mathbf{R}^{n-1}) \oplus \Lambda_{\mathbf{C}}^{p-1}(\mathbf{R}^{n-1})$, we have an isomorphism $j : \Lambda_{\mathbf{C}}^p(\mathbf{R}^{n-1}) \oplus \Lambda_{\mathbf{C}}^{p-1}(\mathbf{R}^{n-1}) \to \Lambda_{\mathbf{C}}^p(\mathbf{R}^n)$ as a vector space and show that the following diagram commutes for any h in $Spin(n-1)$:

$$\begin{array}{ccc} \Lambda_{\mathbf{C}}^p(\mathbf{R}^{n-1}) \oplus \Lambda_{\mathbf{C}}^{p-1}(\mathbf{R}^{n-1}) & \xrightarrow{j} & \Lambda_{\mathbf{C}}^p(\mathbf{R}^n) \\ {\scriptstyle \mathrm{Ad}_{n-1}(h)}\downarrow & & \downarrow{\scriptstyle \mathrm{Ad}_n(h)} \\ \Lambda_{\mathbf{C}}^p(\mathbf{R}^{n-1}) \oplus \Lambda_{\mathbf{C}}^{p-1}(\mathbf{R}^{n-1}) & \xrightarrow{j} & \Lambda_{\mathbf{C}}^p(\mathbf{R}^n). \end{array} \tag{2.6}$$

To prove the case $p = 2s + 1$, we use another isomorphism,

$$e_n j : \mathbf{C}l_{n-1} \ni \psi \mapsto e_n \cdot j(\psi) \in \mathbf{C}l_n^1. \tag{2.7}$$

If we replace j by $e_n j$ on the above diagram (2.6), then we show that it also commutes. □

3. Homogeneous vector bundles on sphere

In this section, we shall describe homogeneous vector bundles on S^{n-1}. We realize S^{n-1} as an orbit space of $Spin(n)$ with base point e_n in $\mathbf{R}^n$. Then S^{n-1} is a homogeneous space

$Spin(n)/Spin(n-1)$ and its spin structure is given by

$$Spin(n) \ni g \mapsto x := g e_n g^{-1} \in S^{n-1} = \frac{Spin(n)}{Spin(n-1)}, \tag{3.1}$$

where the total space is $Spin(n)$.

We shall construct homogeneous vector bundles on S^{n-1}. First, we consider the Clifford bundle $\mathbf{C}l(S^{n-1}) := Spin(n) \times_{\mathrm{Ad}_{n-1}} \mathbf{C}l_{n-1}$. We know that $\mathbf{C}l(S^{n-1})$ is isomorphic to the bundle of differential forms, $\sum A^p_{\mathbf{C}}(S^{n-1})$. Here, $A^p_{\mathbf{C}}(S^{n-1})$ is the bundle of p-forms on S^{n-1}, which is the homogeneous vector bundle corresponding the representation on $\Lambda^p_{\mathbf{C}}(\mathbf{R}^{n-1})$. The sections of $\mathbf{C}l(S^{n-1})$ are given by the $Spin(n-1)$-equivariant functions from $Spin(n)$ to $\mathbf{C}l_{n-1}$:

$$\begin{aligned} C^\infty(\mathbf{C}l(S^{n-1})) &= \{\Psi : Spin(n) \to \mathbf{C}l_{n-1} | \Psi(gh) \\ &= h^{-1}\Psi(g)h \quad \text{for } h \in Spin(n-1)\}. \end{aligned} \tag{3.2}$$

If we define the action of $Spin(n)$ on $C^\infty(\mathbf{C}l(S^{n-1}))$ by $(g_0\Psi)(g) = \Psi(g_0^{-1}g)$ for g_0 in $Spin(n)$, then we obtain a unitary representation of $Spin(n)$ on $L^2(\mathbf{C}l(S^{n-1}))$.

We shall trivialize $\mathbf{C}l(S^{n-1})$. From Lemma 2.1, we know that $\mathbf{C}l(S^{n-1})$ is isomorphic to $Spin(n) \times_{\mathrm{Ad}^i_n} \mathbf{C}l^i_n$ for $i = 0, 1$ as a homogeneous vector bundle. Then, we have the bundle isomorphisms,

$$Spin(n) \times_{\mathrm{Ad}^i_n} \mathbf{C}l^i_n \ni [g, \Psi] \mapsto (g e_n g^{-1}, g\Psi g^{-1}) \in S^{n-1} \times \mathbf{C}l^i_n. \tag{3.3}$$

For $\Psi(g)$ in $C^\infty(\mathbf{C}l(S^{n-1}))$, we define a $\mathbf{C}l^i_n$-valued function $\psi(x)$ on S^{n-1} by $\psi(x) := g\Psi(g)g^{-1}$. Then we regard $C^\infty(\mathbf{C}l(S^{n-1}))$ as the space of the $\mathbf{C}l^i_n$-valued functions on S^{n-1} and see that the action of $Spin(n)$ is given by

$$Spin(n) \times C^\infty(\mathbf{C}l(S^{n-1})) \ni (g_0, \psi(x)) \mapsto g_0\psi(g_0^{-1} x g_0) g_0^{-1} \in C^\infty(\mathbf{C}l(S^{n-1})). \tag{3.4}$$

Remark 3.1. From Lemma 2.1, we can show that

$$A^p_{\mathbf{C}}(S^{n-1}) \oplus A^{p-1}_{\mathbf{C}}(S^{n-1}) \simeq S^{n-1} \times \Lambda^p_{\mathbf{C}}(\mathbf{R}^n). \tag{3.5}$$

Of course, $A^p_{\mathbf{C}}(S^{n-1})$ is not always trivial.

We can also trivialize the spinor bundle $\mathbf{S}(S^{n-1}) := Spin(n) \times_{\Delta_{n-1}} W_{n-1}$ by using (2.3) and (2.4) [6]. It allow us to think of the spinor sections as the spinor-valued functions on S^{n-1}. We see that the action of $Spin(n)$ on the spinor-valued functions is given by

$$Spin(n) \times C^\infty(\mathbf{S}(S^{n-1})) \ni (g_0, \phi(x)) \mapsto g_0\phi(g_0^{-1} x g_0) \in C^\infty(\mathbf{S}(S^{n-1})). \tag{3.6}$$

From the above discussions, we deal with the trivial bundles $S^{n-1} \times \mathbf{C}l_n$ and $S^{n-1} \times W_n$. The space of functions on S^{n-1} is isomorphic to $\sum H^q$ as a representation space of $Spin(n)$, where we denote the space of the harmonic polynomials with polynomial's degree q on $\mathbf{R}^n$ by H^q. Considering the tensor representations on $H^q \otimes \mathbf{C}l^i_n$ and $H^q \otimes W_n$, we see that the actions on these spaces are nothing else but (3.4) and (3.6), respectively.

Proposition 3.2. *We have the following isomorphisms as representation spaces of* $Spin(n)$:

1. *The Clifford case*:

$$L^2(\mathbf{C}l(S^{n-1})) \simeq \sum_{q\geq 0} H^q \otimes \mathbf{C}l_n^i \quad \text{for } i = 0 \text{ or } 1. \tag{3.7}$$

2. *The spinor case*:

$$L^2(\mathbf{S}(S^{2m})) \simeq \sum_{q\geq 0} H^q \otimes W_{2m+1}, \qquad L^2(\mathbf{S}(S^{2m-1})) \simeq \sum_{q\geq 0} H^q \otimes W_{2m}^{\pm}. \tag{3.8}$$

4. Spinor-valued harmonic polynomials

In this section, we give some results for the spinor-valued harmonic polynomials. Let S^q be the space of the polynomials with degree q on $\mathbf{R}^n$. The spaces S^q and H^q have the inner product defined by

$$(f(x), g(x))_S = \left(\sum_\alpha f^\alpha x_\alpha, \sum_\beta g^\beta x_\beta \right)_S := \sum \alpha! f^\alpha \bar{g}^\alpha. \tag{4.1}$$

This inner product satisfies that $(\partial/\partial x_k f, g)_S = (f, x_k g)_S$ for any k. On the other hand, there is an inner product $(\cdot,\cdot)_W$ on W_n such that $(e_k v, w)_W = -(v, e_k w)_W$ for any k. Then we have the inner product $(\cdot,\cdot)$ on $\sum S^q \otimes W_n$ and $\sum H^q \otimes W_n$ such that $(\mathrm{D}\phi, \phi') = -(\phi, x\phi')$. Here D is the Dirac operator on $\mathbf{R}^n$ defined by $\sum e_k \partial/\partial x_k$ and x is the Clifford action by $\sum x_k e_k$.

Trautman [14] considers the following complex $(H^* \otimes W_n, \mathrm{D})$ to analyze $\sum H^q \otimes W_n$:

$$\cdots \xrightarrow{\mathrm{D}} H^{q+1} \otimes W_n \xrightarrow{\mathrm{D}} H^q \otimes W_n \xrightarrow{\mathrm{D}} H^{q-1} \otimes W_n \xrightarrow{\mathrm{D}} \cdots. \tag{4.2}$$

If we have $\mathrm{D}\phi = 0$ for $\phi \in H^q \otimes W_n$, then we show that $\sum \partial^2/\partial x_k^2 (x\phi) = 0$ and $\mathrm{D}(x\phi) = (n-q)\phi$. Therefore, this complex is exact and the space $H^q \otimes W_n$ decomposes as the orthogonal direct sum $\ker^q \mathrm{D} \oplus x(\ker^{q-1} \mathrm{D})$, where $\ker^q \mathrm{D}$ is the kernel of D on $H^q \otimes W_n$. In the following section, we try to apply this method to the Clifford algebra-valued harmonic polynomials.

5. Clifford algebra-valued harmonic polynomials

In this section, we discuss a geometrical decomposition of the $\mathbf{C}l_n$-valued harmonic polynomials on $\mathbf{R}^n$. Because of the isomorphism $\mathbf{C}l_n \simeq \sum \Lambda^p$, we consider $H_p^q := H^q \otimes \Lambda^p$ and $S_p^q := S^q \otimes \Lambda^p$. Here Λ^p denotes $\Lambda^p_{\mathbf{C}}(\mathbf{R}^n)$. We have the following algebraic operators on $\sum \Lambda^p$:

$$e_{k\wedge} : \Lambda^p \to \Lambda^{p+1}, \qquad i(e_k) : \Lambda^p \to \Lambda^{p-1}, \tag{5.1}$$

where $i(e_k)$ is the contraction by e_k. We can easily calculate the relations for $\{e_{k\wedge}\}_k$ and $\{i(e_l)\}_l$,

$$e_{k\wedge}e_{l\wedge} + e_{l\wedge}e_{k\wedge} = 0, \tag{5.2}$$

$$i(e_k)i(e_l) + i(e_l)i(e_k) = 0, \tag{5.3}$$

$$e_{k\wedge}i(e_l) + i(e_l)e_{k\wedge} = \delta_{kl}. \tag{5.4}$$

We define some operators on $\sum S^q \otimes \Lambda^p$ as follows:

$$\mathrm{d} := \sum_{k=1}^{n} \frac{\partial}{\partial x_k} e_{k\wedge} : S_p^q \to S_{p+1}^{q-1}, \tag{5.5}$$

$$\mathrm{d}^* := -\sum_{k=1}^{n} \frac{\partial}{\partial x_k} i(e_k) : S_p^q \to S_{p-1}^{q-1}, \tag{5.6}$$

$$x_\wedge := \sum_{k=1}^{n} x_k e_{k\wedge} : S_p^q \to S_{p+1}^{q+1}, \tag{5.7}$$

$$i(x) := \sum_{k=1}^{n} x_k i(e_k) : S_p^q \to S_{p-1}^{q+1}, \tag{5.8}$$

$$\Box := \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d} = -\sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2} : S_p^q \to S_p^{q-2}. \tag{5.9}$$

We are concerned with the commutation relations among these operators.

Lemma 5.1. *The above operators satisfy the following relations*:

$$\mathrm{d}^2 = \mathrm{d}^{*2} = (x_\wedge)^2 = i(x)^2 = 0, \tag{5.10}$$

$$\mathrm{d}x_\wedge + x_\wedge \mathrm{d} = 0, \qquad \mathrm{d}^* i(x) + i(x)\mathrm{d}^* = 0, \tag{5.11}$$

$$\mathrm{d}i(x) + i(x)\mathrm{d} = r\frac{\partial}{\partial r} + L, \qquad \mathrm{d}^* x_\wedge + x_\wedge \mathrm{d}^* = -r\frac{\partial}{\partial r} - n + L, \tag{5.12}$$

$$\Box x_\wedge - x_\wedge \Box = -2\mathrm{d}, \qquad \Box i(x) - i(x)\Box = 2\mathrm{d}^*, \tag{5.13}$$

$$\Box x_\wedge i(x) - x_\wedge i(x)\Box = 2x_\wedge \mathrm{d}^* + 2i(x)\mathrm{d} - 2r\frac{\partial}{\partial r} - 2L, \tag{5.14}$$

$$\Box i(x)x_\wedge - i(x)x_\wedge \Box = -2x_\wedge \mathrm{d}^* - 2i(x)\mathrm{d} - 2r\frac{\partial}{\partial r} - 2n - 2L, \tag{5.15}$$

$$\Box \mathrm{d} = \mathrm{d}\Box, \qquad \Box \mathrm{d}^* = \mathrm{d}^*\Box. \tag{5.16}$$

Here we set $r := |x|$ and $L := \sum e_{k\wedge} i(e_k)$. The operator $r\partial/\partial r$ (*resp. L*) *measures the polynomial's degree* (*resp. the form's degree*). *In other words, the operator $r\partial/\partial r$* (*resp. L*) *is $q \cdot i\mathrm{d}$* (*resp. $p \cdot i\mathrm{d}$*) *on S_p^q.*

Proof. We remark that $r\partial/\partial r$ is $\sum x_k \partial/\partial x_k$ and prove the lemma straightforwardly. □

Now, there is an inner product $(\cdot,\cdot)_\Lambda$ on $\sum \Lambda^p$ such that $(e_{k\wedge}\psi,\psi')_\Lambda = (\psi, i(e_k)\psi')_\Lambda$ for any k. Then, we obtain the tensor inner product $(\cdot,\cdot)$ on $\sum S^q \otimes \Lambda^p$ satisfying

$$(\mathrm{d}\psi,\psi') = (\psi, i(x)\psi'), \qquad (\mathrm{d}^*\psi,\psi') = -(\psi, x_\wedge \psi'). \tag{5.17}$$

These relations imply that the kernel of d (resp. d^*) is orthogonal to the image of $i(x)$ (resp. $x_\wedge$).

We shall investigate the complexes (H_*^{q-*},d) and $(H_{n-*}^{q-*},\mathrm{d}^*)$, i.e.,

$$(H_*^{q-*},\mathrm{d}) : 0 \xrightarrow{\mathrm{d}} H_0^q \xrightarrow{\mathrm{d}} H_1^{q-1} \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} H_n^{q-n} \xrightarrow{\mathrm{d}} 0, \tag{5.18}$$

$$(H_{n-*}^{q-*},\mathrm{d}^*) : 0 \xrightarrow{\mathrm{d}^*} H_n^q \xrightarrow{\mathrm{d}^*} H_{n-1}^{q-1} \xrightarrow{\mathrm{d}^*} \cdots \xrightarrow{\mathrm{d}^*} H_0^{q-n} \xrightarrow{\mathrm{d}^*} 0. \tag{5.19}$$

If we have ψ in H_p^{q-p} such that $\mathrm{d}\psi = 0$, then we have $\mathrm{d}i(x)\psi = q\psi$ from (5.12). But $i(x)\psi$ is not necessarily harmonic because $\Box i(x)\psi = (i(x)\Box + \mathrm{d}^*)\psi = \mathrm{d}^*\psi$. Thus, to prove the exactness of these complex is more complicated than the case of the spinors. So, we discuss the following complexes for $\{S_p^q\}$ instead of $\{H_p^q\}$:

$$(S_*^{q-*},\mathrm{d}) : 0 \xrightarrow{\mathrm{d}} S_0^q \xrightarrow{\mathrm{d}} S_1^{q-1} \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} S_n^{q-n} \xrightarrow{\mathrm{d}} 0, \tag{5.20}$$

$$(S_{n-*}^{q-*},\mathrm{d}^*) : 0 \xrightarrow{\mathrm{d}^*} S_n^q \xrightarrow{\mathrm{d}^*} S_{n-1}^{q-1} \xrightarrow{\mathrm{d}^*} \cdots \xrightarrow{\mathrm{d}^*} S_0^{q-n} \xrightarrow{\mathrm{d}^*} 0, \tag{5.21}$$

$$(S_*^{q+*},x_\wedge) : 0 \xrightarrow{x_\wedge} S_0^q \xrightarrow{x_\wedge} S_1^{q+1} \xrightarrow{x_\wedge} \cdots \xrightarrow{x_\wedge} S_n^{q+n} \xrightarrow{x_\wedge} 0, \tag{5.22}$$

$$(S_{n-*}^{q+*},i(x)) : 0 \xrightarrow{i(x)} S_n^q \xrightarrow{i(x)} S_{n-1}^{q+1} \xrightarrow{i(x)} \cdots \xrightarrow{i(x)} S_0^{q+n} \xrightarrow{i(x)} 0. \tag{5.23}$$

Proposition 5.2. *The complexes* (S_*^{q-*},d), $(S_{n-*}^{q-*},\mathrm{d}^*)$, $(S_*^{q+*},x_\wedge)$, *and* $(S_{n-*}^{q+*},i(x))$ *are exact. It follows that*

$$\dim \ker_p^q \mathrm{d} = \binom{n+q}{p+q}\binom{p+q-1}{p-1} \quad \text{for } p \neq 0, \tag{5.24}$$

$$\dim \ker_p^q \mathrm{d}^* = \binom{n+q}{p}\binom{n+q-p+1}{q} \quad \text{for } p \neq n, \tag{5.25}$$

$$\dim \ker_0^q \mathrm{d} = \dim \ker_n^q \mathrm{d}^* = 1, \tag{5.26}$$

where $\ker_q^p \mathrm{d}$ (*resp.* $\ker_q^p \mathrm{d}^*$) *is the kernel of* d (*resp.* d^*) *on* S_q^p.

Proof. The exactness follows from (5.12). Then we can calculate dimensions of $\ker_q^p \mathrm{d}$ and $\ker_q^p \mathrm{d}^*$. For example, we have

$$\dim \ker_p^q \mathrm{d} = \sum_{m=0}^{p-1} (-1)^{m-(p-1)} \dim S_m^{p+q-m}, \tag{5.27}$$

where

$$\dim S_p^q = \dim S^q \otimes \Lambda^q = \binom{n+q-1}{q}\binom{n}{p}.$$

By the induction for p, we obtain (5.24). □

Corollary 5.3. *We have decompositions of* S_p^q *as follows*:

$$S_p^q = \ker_p^q \mathrm{d} \oplus \ker_p^q i(x), \tag{5.28}$$

$$S_p^q = \ker_p^q \mathrm{d}^* \oplus \ker_p^q x_\wedge, \tag{5.29}$$

$$S_p^q = \ker_p^q \mathrm{d} \cap \ker_p^q \mathrm{d}^* \oplus \ker_p^q x_\wedge \oplus \ker_p^q i(x), \tag{5.30}$$

where $\ker_p^q \mathrm{d}$ (*resp.* $\ker_p^q \mathrm{d}^*$) *is orthogonal to* $\ker_p^q i(x)$ (*resp.* $\ker_p^q x_\wedge$). *The dimension of* $\ker_p^q \mathrm{d} \cap \ker_p^q \mathrm{d}^*$ *is given by*

$$\dim \ker_p^q \mathrm{d} \cap \ker_p^q \mathrm{d}^* = \frac{(n+q-1)!}{(n-p-1)!(p-1)!q!}\frac{n+2q}{(p+q)(n+q-p)}. \tag{5.31}$$

Proof. From Proposition 5.2, we have $\ker_p^q i(x) = \mathrm{Im}_{p+1}^{q-1} i(x)$. So $\ker_p^q \mathrm{d}$ is orthogonal to $\ker_p^q i(x)$. Besides, we have $\mathrm{d}i(x)\phi + i(x)\mathrm{d}\phi = (p+q)\phi$ for ψ in S_p^q and conclude that S_p^q decomposes as the orthogonal direct sum of $\ker_p^q \mathrm{d}$ and $\ker_p^q i(x)$. Similarly, we have the second decomposition (5.29). To show the third decomposition, we consider the orthogonal complement of $\ker_p^q \mathrm{d} \cap \ker_p^q \mathrm{d}^*$:

$$(\ker_p^q \mathrm{d} \cap \ker_p^q \mathrm{d}^*)^\perp = (\ker_p^q \mathrm{d})^\perp + (\ker_p^q \mathrm{d}^*)^\perp = \ker_p^q i(x) + \ker_p^q x_\wedge. \tag{5.32}$$

We shall prove that $\ker_p^q i(x) \cap \ker_p^q x_\wedge$ is zero. We have the following relation between $x_\wedge$ and $i(x)$:

$$x_\wedge i(x) + i(x)x_\wedge = |x|^2 = r^2. \tag{5.33}$$

Since the multiplication r^2 is injective on S_p^q, we show that $\ker_p^q i(x) \cap \ker_p^q x_\wedge = 0$ and hence $\ker_p^q i(x) + \ker_p^q x_\wedge = \ker_p^q i(x) \oplus \ker_p^q x_\wedge$. Then, we have the third decomposition (5.30). The decomposition (5.30) allows us to calculate the dimension of $\ker_p^q \mathrm{d} \cap \ker_p^q \mathrm{d}^*$:

$$\dim \ker_p^q \mathrm{d} \cap \ker_p^q \mathrm{d}^* = \dim \ker_p^q \mathrm{d} + \dim \ker_p^q \mathrm{d}^* - \dim S_p^q. \tag{5.34}$$

From (5.24) and (5.25), we obtain $\dim \ker_p^q \mathrm{d} \cap \ker_p^q \mathrm{d}^*$ as (5.31). □

Remark 5.4. The vector space $\ker_p^q i(x)$ is not always orthogonal to $\ker_p^q x_\wedge$.

We denote the vector space $\ker_p^q \mathrm{d} \cap \ker_p^q \mathrm{d}^*$ by I_p^q, which is a subspace of H_p^q. We shall decompose the vector spaces $\ker_p^q x_\wedge$ and $\ker_p^q i(x)$ further. The exactness of $(S_*^{q+*}, x_\wedge)$

means that $\ker_p^q x_\wedge = x_\wedge(S_{p-1}^{q-1})$. Then, from Corollary 5.3, we have

$$\begin{aligned}\ker_p^q x_\wedge &= x_\wedge(S_{p-1}^{q-1}) = x_\wedge(I_{p-1}^{q-1}) \oplus x_\wedge(\ker_{p+1}^{q-1} x_\wedge) \oplus x_\wedge(\ker_{p-1}^{q-1} i(x))\\ &= x_\wedge I_{p-1}^{q-1} \oplus x_\wedge(\ker_{p-1}^{q-1} i(x)),\end{aligned} \tag{5.35}$$

where we remark that the map $x_\wedge$ is injective on I_{p-1}^{q-1} and $\ker_{p-1}^{q-1} i(x)$. In the same way, we have

$$\ker_p^q i(x) = i(x) I_{p+1}^{q-1} \oplus i(x)(\ker_{p+1}^{q-1} x_\wedge), \tag{5.36}$$

and know that the map $i(x)$ is injective on I_{p+1}^{q-1} and $\ker_{p+1}^{q-1} x_\wedge$. It is easy to see that $x_\wedge I_{p-1}^{q-1}$ and $i(x) I_{p+1}^{q-1}$ are subspaces of H_p^q and orthogonal to each other.

Since we have shown that $I_p^q \oplus x_\wedge I_{p-1}^{q-1} \oplus i(x) I_{p+1}^{q-1}$ is in H_p^q, we consider the direct sum of the remaining parts $x_\wedge(\ker_{p-1}^{q-1} i(x))$ and $i(x)(\ker_{p+1}^{q-1} x_\wedge)$. Here,

$$x_\wedge(\ker_{p-1}^{q-1} i(x)) = x_\wedge i(x) I_p^{q-2} \oplus x_\wedge i(x)(\ker_p^{q-2} x_\wedge), \tag{5.37}$$

$$i(x)(\ker_{p+1}^{q-1} x_\wedge) = i(x) x_\wedge I_p^{q-2} \oplus i(x) x_\wedge(\ker_p^{q-2} i(x)). \tag{5.38}$$

To get the harmonic part of $x_\wedge(\ker_{p-1}^{q-1} i(x)) \oplus i(x)(\ker_{p+1}^{q-1} x_\wedge)$, we use the decomposition of S_p^q into harmonic part H_p^q and its orthogonal complement $r^2 S_p^{q-2}$, i.e., $S_p^q = H_p^q \oplus r^2 S_p^{q-2}$. The complement part $r^2 S_p^{q-2}$ has the decomposition

$$\begin{aligned}r^2 S_p^{q-2} &= (x_\wedge i(x) + i(x) x_\wedge) S_p^{q-2}\\ &= r^2 I_p^{q-2} \oplus x_\wedge i(x)(\ker_p^{q-2} x_\wedge) \oplus i(x) x_\wedge(\ker_p^{q-2} i(x)).\end{aligned} \tag{5.39}$$

On the other hands, we have had the following decomposition of S_p^q:

$$\begin{aligned}S_p^q = {}& I_p^q \oplus x_\wedge I_{p-1}^{q-1} \oplus i(x) I_{p+1}^{q-1} \oplus x_\wedge i(x) I_p^{q-2} \oplus i(x) x_\wedge I_p^{q-2}\\ &\oplus x_\wedge i(x)(\ker_p^{q-2} x_\wedge) \oplus i(x) x_\wedge(\ker_p^{q-2} i(x)).\end{aligned} \tag{5.40}$$

Comparing the above two decompositions, we remark that $x_\wedge i(x) I_p^{q-2} \oplus i(x) x_\wedge I_p^{q-2}$ can be decomposed into the harmonic and the non-harmonic parts.

Lemma 5.5. *The vector space $x_\wedge i(x) I_p^{q-2} \oplus i(x) x_\wedge I_p^{q-2}$ has the decomposition*

$$x_\wedge i(x) I_p^{q-2} \oplus i(x) x_\wedge I_p^{q-2} = h_p^{q-2} I_p^{q-2} \oplus r^2 I_p^{q-2}. \tag{5.41}$$

Here, the map h_p^q is defined by

$$h_p^q := (q+n-p) x_\wedge i(x) - (q+p) i(x) x_\wedge : S_p^q \to S_p^{q+2}. \tag{5.42}$$

This map h_p^q is injective on I_p^q and its image $h_p^q I_p^q$ is in H_p^{q+2}.

Proof. Since the maps $i(x)x_\wedge$, $i(x)x_\wedge$, and r^2 are injective on I_p^q, we have

$$x_\wedge i(x)I_p^{q-2} \oplus i(x)x_\wedge I_p^{q-2} = r^2 I_p^{q-2} \oplus (ax_\wedge i(x) + bi(x)x_\wedge)I_p^{q-2}, \tag{5.43}$$

where we choose a pair of constants (a, b) such that $(a, b) \neq \lambda(1, 1)$ for any λ in $\mathbf{C}$. If we put $(a, b) = (q + n - p - 2, -(q + p - 2))$, then we show from (5.14) and (5.15) that $(ai(x)x_\wedge + bx_\wedge i(x))(I_p^{q-2})$ is harmonic. □

We are now in a position to describe a decomposition of the Clifford algebra-valued or the exterior algebra-valued harmonic polynomials.

Theorem 5.6. *The space of the exterior algebra-valued harmonic polynomials with polynomial's degree q and form's degree p is decomposed as follows*:

$$\begin{aligned} H_p^q &= \ker_p^q \mathrm{d} \cap \ker_p^q \mathrm{d}^* \oplus x_\wedge(\ker_{p-1}^{q-1} \mathrm{d} \cap \ker_{p-1}^{q-1} \mathrm{d}^*) \oplus i(x)(\ker_{p+1}^{q-1} \mathrm{d} \cap \ker_{p+1}^{q-1} \mathrm{d}^*) \\ &\quad \oplus h_p^{q-2}(\ker_p^{q-2} \mathrm{d} \cap \ker_p^{q-2} \mathrm{d}^*), \end{aligned} \tag{5.44}$$

where each component is orthogonal to others. Furthermore, we have

$$\ker_p^q \mathrm{d} \cap H_p^q = \ker_p^q \mathrm{d} \cap \ker_p^q \mathrm{d}^* \oplus x_\wedge(\ker_{p-1}^{q-1} \mathrm{d} \cap \ker_{p-1}^{q-1} \mathrm{d}^*), \tag{5.45}$$

$$\ker_p^q \mathrm{d}^* \cap H_p^q = \ker_p^q \mathrm{d} \cap \ker_p^q \mathrm{d}^* \oplus i(x)(\ker_{p+1}^{q-1} \mathrm{d} \cap \ker_{p+1}^{q-1} \mathrm{d}^*). \tag{5.46}$$

Proof. The orthogonal decomposition of H_p^q follows from discussions in this section. So we shall prove (5.45) and (5.46). It is clear that $\ker_p^q \mathrm{d} \cap H_p^q$ has the subspace $I_p^q \oplus x_\wedge I_{p-1}^{q-1}$. For ϕ in $\ker_p^q \mathrm{d} \cap H_p^q$, we have $\mathrm{d}^* x_\wedge \phi + x_\wedge \mathrm{d}^* \phi = (-q - n + p)\phi$ and show that $\mathrm{d}^* x_\wedge \phi$ is in I_p^q and $x_\wedge \mathrm{d}^* \phi$ is in $x_\wedge I_{p-1}^{q-1}$. Thus, we have $\ker_p^q \mathrm{d} \cap H_p^q = I_p^q \oplus x_\wedge I_{p+1}^{q-1}$. Similarly, we can prove (5.46). □

This theorem implies the exactness of the complexes (H_*^{q-*}, d) and $(H_{n-*}^{q-*}, \mathrm{d}^*)$.

Corollary 5.7 (Poincaré lemma for harmonic polynomials on $\mathbf{R}^n$). *The complexes* (H_*^{q-*}, d) *and* $(H_{n-*}^{q-*}, \mathrm{d}^*)$ *are exact*.

Proof. From Theorem 5.6, it follows that

$$\ker_p^q \mathrm{d} \cap H_p^q = I_p^q \oplus x_\wedge I_{p-1}^{q-1}, \tag{5.47}$$

$$\mathrm{d}(H_{p-1}^{q+1}) = \mathrm{d}(i(x)I_p^q) \oplus \mathrm{d}(h_{p-1}^{q-1} I_{p-1}^{q-1}), \tag{5.48}$$

where the map d is injective on $i(x)I_p^q$ and $h_{p-1}^{q-1} I_{p-1}^{q-1}$. We show that $\mathrm{d}(i(x)I_q^p)$ is a subspace in I_p^q and $\dim I_p^q = \dim \mathrm{d}(i(x)I_p^q)$. Therefore, the map $\mathrm{d} : i(x)I_p^q \to I_p^q$ is isomorphism. In the same way, we show that the map $\mathrm{d} : h_{p-1}^{q-1} I_{p-1}^{q-1} \to x_\wedge I_{p-1}^{q-1}$ is isomorphism. Then, we conclude that (H_*^{q-*}, d) is exact. Similarly, we prove that $(H_{n-*}^{q-*}, \mathrm{d}^*)$ is exact. □

6. Some representations of *Spin*(n)

In this section, we present some results for the representation theory of $Spin(n)$ by using the Clifford algebra [9,11–13]. Let $\mathfrak{g}$ be the Lie algebra $\mathfrak{spin}(n) = \mathbf{R}\{e_k e_l\}_{k<l}$ in $\mathbf{C}l_n$ with bracket $[a, b] = ab - ba$ and let $\mathfrak{g}_\mathbf{C}$ be the complexification of $\mathfrak{g}$, i.e., $\mathfrak{g}_\mathbf{C} = \mathfrak{g} \otimes \mathbf{C}$. Since all the finite dimensional complex irreducible representations of $Spin(n)$ correspond to the ones of $\mathfrak{g}_\mathbf{C}$, we investigate the representations of $\mathfrak{g}_\mathbf{C}$.

Definition 6.1. For $1 \le k \le [\frac{1}{2}n]$,

$$a_k := \tfrac{1}{2}(\sqrt{-1}e_{2k-1} - e_{2k}), \qquad a_k^\dagger := \tfrac{1}{2}(\sqrt{-1}e_{2k-1} + e_{2k}), \tag{6.1}$$

$$\omega_k := a_k^\dagger a_k - \frac{1}{2} = -\frac{\sqrt{-1}}{2} e_{2k-1}e_{2k}, \tag{6.2}$$

$$z_k := x_{2k-1} + \sqrt{-1}x_{2k}, \qquad \bar{z}_k := x_{2k-1} - \sqrt{-1}x_{2k}. \tag{6.3}$$

When $n = 2m + 1$,

$$b := \sqrt{-1}e_{2m+1}. \tag{6.4}$$

We put $[a, b]_+ := ab + ba$ and rewrite the Clifford relation (2.1) as follows:

$$[a_k, a_l^\dagger]_+ = \delta_{kl}, \qquad [b, b]_+ = 2, \tag{6.5}$$

$$[a_k, a_l]_+ = [a_k^\dagger, a_l^\dagger]_+ = [a_k, b]_+ = [a_k^\dagger, b]_+ = 0. \tag{6.6}$$

We choose the sub-algebra $\mathbf{R}\{\sqrt{-1}\omega_k\}_k$ as a Cartan sub-algebra of $\mathfrak{g}$ and define a dual basis $\{f_k\}_k$ of $\{\omega_k\}_k$ by $f_l(\omega_k) = \delta_{kl}$. The irreducible finite dimensional representations of $\mathfrak{g}_\mathbf{C}$ are parameterized by the dominant integral weights. The weight $\lambda = \sum_{k=1}^m s_k f_k$ is dominant integral if and only if $s = (s_1, \dots, s_m)$ satisfies that

$$s_1 \ge \cdots s_{m-1} \ge |s_m|, \quad n = 2m, \tag{6.7}$$

$$s_1 \ge \cdots s_{m-1} \ge s_m \ge 0, \quad n = 2m + 1, \tag{6.8}$$

where s is in $\mathbf{Z}^m$ or $\mathbf{Z}^m + (\frac{1}{2}, \dots, \frac{1}{2})$. We denote the weight $\lambda = \sum s_k f_k$ by $s = (s_1, \dots, s_m)$, and a string of j k's for k in $\mathbf{Z} \cup \frac{1}{2}\mathbf{Z}$ by k_j. For example, $((\frac{3}{2})_p, (\frac{1}{2})_{m-p})$ is the weight whose first p components are $\frac{3}{2}$ and others are $\frac{1}{2}$. Besides, we denote the representation space corresponding to $s = (s_1, \dots, s_m)$ by $V(s_1, \dots, s_m)$.

We shall present some representations of $\mathfrak{g}$. First, the space of harmonic polynomials H^q gives the irreducible representation space whose highest weight vector is $\bar{z}_1^q$ with weight $(q, 0_{m-1})$.

Next, the spinor space W_n is given by $\{a_{k_1}^\dagger \cdots a_{k_j}^\dagger |vac\rangle | 1 \le k_1 < \cdots < k_j \le m\}$, where we define $a_k|vac\rangle := 0$ for any k and $b|vac\rangle := |vac\rangle$. Then W_{2m}^+ (resp. W_{2m}^-) gives the irreducible representation space whose highest vector is $a_1^\dagger \cdots a_m^\dagger |vac\rangle$ with weight $((\frac{1}{2})_m)$

(resp. $a_1^\dagger \cdots a_{m-1}^\dagger |vac\rangle$ with weight $((\frac{1}{2})_{m-1}, -\frac{1}{2}))$ and W_{2m+1} does the one whose highest weight vector is $a_1^\dagger \cdots a_m^\dagger |vac\rangle$ with weight $((\frac{1}{2})_m)$.

Finally, we consider the space of p-forms, Λ^p. Under the isomorphism $\sum \Lambda^p = \mathbf{C}l_n$, the action of $\mathfrak{spin}(n)$ is defined by $\mathrm{ad}(e_k e_l)(\phi) = e_k e_l \phi - \phi e_k e_l$ for ϕ in Λ^p. We define the algebraic operator ω commuting with the action of $\mathfrak{spin}(n)$ by

$$\omega : \Lambda^p \ni \psi \mapsto 2^m \omega_1 \cdots \omega_m \psi \in \Lambda^{n-p}. \tag{6.9}$$

This operator is called the complex volume element and satisfies that $\omega^2 = 1$ and $e_k \omega = -\omega e_k$. We know that, for $0 \leq p \leq m$ except the case of $n = 2m$ and $p = m$, Λ^p is equivalent to Λ^{n-p} by the operator ω and gives the irreducible representation space whose highest weight vector is $a_1^\dagger \cdots a_p^\dagger$ with weight $(1_p, 0_{m-p})$. For $n = 2m$ and $p = m$, ω decomposes Λ^m into ± 1-eigenspace $\Lambda^m_\pm$. Then Λ^m_+ (resp. Λ^m_-) has the highest weight vector $a_1^\dagger \cdots a_m^\dagger$ with weight (1_m) (resp. $a_1^\dagger \cdots a_{m-1}^\dagger a_m$ with weight $(1_{m-1}, -1)$).

7. The irreducible decomposition of $H^q \otimes \mathbf{C}l_n$

In this section, we show that our geometrical decompositions of $H^q \otimes \Lambda^p$ are the irreducible decompositions with respect to $Spin(n)$. For the spinor case, we can prove similar results [2,3,8,14].

The actions of $\mathfrak{g}_{\mathbf{C}}$ on $H^q \otimes \Lambda^p$ are given by

$$\mathfrak{g}_{\mathbf{C}} \times (H^q \otimes \Lambda^q) \ni \left(\frac{e_k e_l}{2}, \psi(x)\right) \mapsto -x_k \frac{\partial \psi}{\partial x_l} + x_l \frac{\partial \psi}{\partial x_k} + \mathrm{ad}\left(\frac{e_k e_l}{2}\right) \psi \in H^q \otimes \Lambda^q. \tag{7.1}$$

We see the following commutation relations between the above actions and the operators given in Section 5.

Lemma 7.1. *On* $\sum H^q \otimes \Lambda^p$, *the operators* d, d*, $x_\wedge$, *and* $i(x)$ *commute with the action of* $\mathfrak{g}_{\mathbf{C}}$. *It follows that each component of the geometrical decomposition* (5.44) *is an invariant subspace for* $\mathfrak{g}_{\mathbf{C}}$.

Proof. We can prove the lemma by straightforward calculations. So we omit the proof. □

In general, if we have two irreducible highest weight representations V_λ and $V_{\lambda'}$ whose highest weight vectors are v_λ and $v_{\lambda'}$, respectively, then we find an irreducible representation $V_{\lambda+\lambda'}$ with highest vector $v_\lambda \otimes v_{\lambda'}$ in $V_\lambda \otimes V_{\lambda'}$. We apply this fact to $H^q \otimes \Lambda^p$.

Since $H^q = V(q, 0_{m-1})$ and $\Lambda^p = V(1_p, 0_{m-p})$, we know that $V(q+1, 1_{p-1}, 0_{m-p})$ is an irreducible component of $H^q \otimes \Lambda^p$, whose highest weight vector is $\psi_0(x) := \bar{z}_1^q a_1^\dagger \cdots a_p^\dagger$. If we show that the vector $\psi_0(x)$ is in I_p^q, then we conclude that $V(q+1, 1_{p-1}, 0_{m-p})$ is a subspace of I_p^q. So we need the following formula of the operators d and d*.

Lemma 7.2. *When we use the notation of Definition* 6.1, *we rewrite the operator* d *and* d^* *on* $H^q \otimes \Lambda^p$ *as follows*:

1. *For* $n = 2m$,

$$\mathrm{d} + \mathrm{d}^* = -2\sqrt{-1}\sum_{k=1}^{m}\left({a_k}^L \frac{\partial}{\partial z_k} + a_k^{\dagger L}\frac{\partial}{\partial \bar{z}_k}\right), \tag{7.2}$$

$$(-1)^p(\mathrm{d} - \mathrm{d}^*) = -2\sqrt{-1}\sum_{k=1}^{m}\left({a_k}^R \frac{\partial}{\partial z_k} + a_k^{\dagger R}\frac{\partial}{\partial \bar{z}_k}\right), \tag{7.3}$$

where ${a_k}^L$ (resp. ${a_k}^R$) is defined by ${a_k}^L\psi := a_k \cdot \psi$ (resp. ${a_k}^R\psi := \psi \cdot a_k$).

2. *For* $n = 2m + 1$, *we add*

$$-\sqrt{-1}b^L\frac{\partial}{\partial x_{2m+1}}, \qquad -\sqrt{-1}b^R\frac{\partial}{\partial x_{2m+1}} \tag{7.4}$$

to the right-hand sides of (7.2) and (7.3), *respectively*.

Proof. We know that ${e_i}^L = e_{i\wedge} - i(e_i)$ and ${e_i}^R = (-1)^p(e_{i\wedge} + i(e_i))$ and prove the lemma. □

From this lemma, we can easily show that $\mathrm{d}(\psi_0(x)) = \mathrm{d}^*(\psi_0(x)) = 0$ and hence $V(q+1, 1_{p-1}, 0_{m-p})$ is in I^q_p. Furthermore, by Weyl's dimension formula, we have $\dim V(q+1, 1_{p-1}, 0_{m-p}) = \dim I^q_p$. Then, we conclude that I^q_p gives the irreducible representation whose highest weight vector is $\psi_0(x)$ with weight $(q+1, 1_{p-1}, 0_{m-p})$.

Proposition 7.3. *The vector space* $I^q_p = \ker^q_p \mathrm{d} \cap \ker^q_p \mathrm{d}^*$ *has the following description as a representation space of* $Spin(n)$.

1. *When* $1 \le p \le [\frac{1}{2}n]$ *or when* $n = 2m + 1$ *and* $p = m$,

$$\ker^q_p \mathrm{d} \cap \ker^q_p \mathrm{d}^* \simeq \ker^q_{n-p} \mathrm{d} \cap \ker^q_{n-p} \mathrm{d}^* \simeq V(q+1, 1_{p-1}, 0_{m-p}). \tag{7.5}$$

2. *When* $n = 2m$,

$$\ker^q_m \mathrm{d} \cap \ker^q_m \mathrm{d}^* \simeq V(q+1, 1_{m-1}) \oplus V(q+1, 1_{m-2}, -1). \tag{7.6}$$

Now, Lemma 7.1 implies the isomorphisms as representation spaces,

$$x_\wedge I^q_p \simeq i(x) I^q_p \simeq h^q_p I^q_p \simeq I^q_p. \tag{7.7}$$

Therefore, we give a representation theoretical meaning to our geometrical decompositions of $H^q \otimes \Lambda^p$.

Corollary 7.4. *We decompose* $H^q \otimes \Lambda^p$ *into irreducible components as follows*:

1. *When* $n = 2m$ *and* $0 \le p \le m - 2$ *or when* $n = 2m + 1$ *and* $0 \le p \le m - 1$,

$$\begin{aligned} H^q \otimes \Lambda^p \simeq H^q \otimes \Lambda^{n-p} &\simeq V(q, 0_{m-1}) \otimes V(1_p, 0_{m-p}) \\ &\simeq V(q+1, 1_{p-1}, 0_{m-p}) \oplus V(q, 1_{p-2}, 0_{m-p+1}) \\ &\quad \oplus V(q, 1_p, 0_{m-p-1}) \oplus V(q-1, 1_{p-1}, 0_{m-p}). \end{aligned} \tag{7.8}$$

2. *When* $n = 2m$,

$$\begin{aligned} H^q \otimes \Lambda^{m-1} \simeq H^q \otimes \Lambda^{m+1} &\simeq V(q, 0_{m-1}) \otimes V(1_{m-1}, 0) \\ &\simeq V(q+1, 1_{m-2}, 0) \oplus (V(q, 1_{m-1}) \oplus V(q, 1_{m-2}, -1)) \\ &\quad \oplus V(q, 1_{m-3}, 0, 0) \oplus V(q-1, 1_{m-2}, 0), \end{aligned} \tag{7.9}$$

$$\begin{aligned} H^q \otimes \Lambda^m &\simeq V(q, 0_{m-1}) \otimes (V(1_m) \oplus V(1_{m-1}, -1)) \\ &\simeq (V(q+1, 1_{m-1}) \oplus V(q+1, 1_{m-2}, -1)) \oplus V(q, 1_{m-2}, 0) \\ &\quad \oplus V(q, 1_{m-2}, 0) \oplus (V(q-1, 1_{m-1}) \oplus V(q-1, 1_{m-2}, -1)). \end{aligned} \tag{7.10}$$

3. *When* $n = 2m + 1$,

$$\begin{aligned} H^q \otimes \Lambda^m &\simeq V(q, 0_{m-1}) \otimes V(1_m) \simeq V(q+1, 1_{m-1}) \oplus V(q, 1_{m-2}, 0) \\ &\quad \oplus V(q, 1_{m-1}) \oplus V(q-1, 1_{m-1}). \end{aligned} \tag{7.11}$$

Remark 7.5. By using the complex volume element ω, we decompose $H^q \otimes \Lambda^m_\pm$ for $n = 2m$:

$$\begin{aligned} H^q \otimes \Lambda^m_\pm &\simeq V(q, 0_{m-1}) \otimes V(1_{m-1}, \pm 1) \\ &\simeq V(q+1, 1_{m-2}, \pm 1) \oplus V(q, 1_{m-2}, 0) \oplus V(q-1, 1_{m-2}, \mp 1). \end{aligned} \tag{7.12}$$

Remark 7.6. From the above corollary and Proposition 3.2, we decompose $L^2(A^p_{\mathbf{C}}(S^{n-1}))$ into irreducible components [9,11].

Acknowledgements

The author would like to thank Prof. T. Kori and Prof. K. Fujii for their good advice and encouragement, and thank Prof. A. Ikeda for letting him know the paper [11]. The author also thanks the referees for their good advice and suggestions.

References

[1] M.F. Atiyah, R. Bott, A. Shapiro, Clifford modules, Topology 3 (1964) 3–38.
[2] C. Bär, The Dirac operator on homogeneous spaces and its spectrum on three-dimensional lens spaces, Arch. Math. 59 (1992) 65–79.
[3] C. Bär, The Dirac operator on space forms of positive curvature, J. Math. Soc. Jpn. 48 (1996) 69–83.
[4] T. Branson, Stein–Weiss operators and ellipticity, J. Funct. Anal. 151 (1997) 334–383.

[5] J. Bureš, V. Souček, Eigenvalues of conformally invariant operators on spheres, in: The 18th Winter School "Geometry and Physics", Rend. Circ. Mat. Palermo 2 (Suppl. 59) (1999) 109–122 (math. DG/9807050).
[6] J. Bureš, Dirac operators on hypersurfaces, Commun. Math. Univ. Carolinae 34 (2) (1993) 313–322.
[7] J. Bureš, The higher spin Dirac operators, Differential Geometry and Applications (Brno. 1998), Masaryk University, Brno, 1999, pp. 319–334 (math. DG/9901039).
[8] R. Delanghe, F. Sommen, V. Souček, Clifford Algebra and Spinor-Valued Functions, Kluwer Academic Publishers, Dordrecht, 1992.
[9] G.B. Folland, Harmonic analysis of the de Rham complex on the sphere, J. Reine Angew. Math. 398 (1989) 130–143.
[10] Y. Homma, A representation of $Spin(4)$ on the eigenspinors of the Dirac operator on S^3, Tokyo J. Math., in press.
[11] A. Ikeda, Y. Taniguchi, Spectra and eigenforms of the Laplacian on S^n and $P^n(\mathbf{C})$, Osaka J. Math. 15 (3) (1978) 515–546.
[12] A.W. Knapp, Representation Theory of Semisimple Groups, Princeton Math. Ser. 36, Princeton Univ. Press, Princeton, NJ, 1986.
[13] H.B. Lawson, M.L. Michelsohn, Spin Geometry, Princeton University Press, Princeton, NJ, 1989.
[14] A. Trautman, The Dirac operator on hypersurfaces, Acta Phys. Polonica B 26 (7) (1995) 1283–1310.